M. Math. First Year Second Semestral Exam
Complex Analysis
April 22, 2024
Instructor — B. Sury
Answer SIX questions INCLUDING Question 7.

Q 1. Let f be an analytic function, and for each $z_0 \in \mathbb{C}$, suppose that the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)$$

has the property that some $c_n = 0$. Prove that f must be a polynomial.
Hint. Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

OR

Q 1. If a_n are non-negative real numbers and the series $\sum_{n\geq 0} a_n z^n$ has radius of convergence 1, then prove that $z = 1$ is a singular point. That is, prove that for any open disc D around 1, there is no analytic function on D which coincides with the above power series in $D \cap B(0,1)$.

Q 2. Let f be entire, and let U be an open bounded set. Show that if $|f|$ is constant on ∂U, then f is either a constant or has a zero in U.

OR

Q 2. Let f be non-constant and holomorphic in an open set containing the closed unit disc. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

Q 3.
(a) Let f be a non-constant, entire function. Show that there exists z_0 such that $f(z_0) \in B(0,1)$.
(b) Let f be analytic on $B(0,2)$. Prove that for $a, b \in B(0,1)$, we have

$$\frac{f(b) - f(a)}{b - a} - f'(0) = \frac{1}{2i\pi} \int_{C(0,1)} \left(\frac{1}{(z-a)(z-b)} - \frac{1}{z^2} \right) f(z) dz.$$

OR

Q 3. Prove that there does not exist a sequence of polynomials $\{f_n\}$ such that $f_n(z) \to 1/z^2$ uniformly on the annulus $A(1,2) := \{z : 1 < |z| < 2\}$.
Hint. Get N so that $|f_n(z) - 1/z^2| < 1/4$ for all $n > N$ and $z \in A(1,2)$. Apply maximum modulus to $z^2 f_N(z) - 1$.

Q 4. Let D be a region and let $f : D \to \mathbb{C}$ be analytic such that $f'(z) \neq 0$ for all $z \in D$. Suppose $z_0 \in D$ satisfies $f(z_0) \neq 0$. Given $\epsilon > 0$, prove that there exist $z_1, z_2 \in B(z_0, \epsilon)$ such that $|f(z_1)| < |f(z_0)| < |f(z_2)|$.

OR

Q 4. Let f be analytic on the punctured unit disc $D^\circ := \{z : 0 < |z| < 1\}$ such that $|f(z) \leq \log(1/|z|)$ for $z \in D^\circ$. Prove that f must be the zero function.
Hint. Show that 0 is a removable singularity and continue.

Q 5. Let $f : B(0,1) \to B(0,1)$ be analytic and suppose $f(0) = 0$. Define the sequence of functions $f^{(n)}$ by $f^{(2)}(z) = f(f(z)), f^{(n+1)}(z) = f(f^{(n)}(z))$ for all $n > 1$. If $f^{(n)}(z) \to g(z)$ for all $z \in B(0,1)$, prove that either g is the zero function or the identity function.
Hint. Apply Schwarz. When $|f(z)/z| < 1$ for all $z \in B(0,1)$, consider for arbitrary $0 < t < 1$, the maximum of $|f(z)/z|$ over $\overline{B(0,t)}$ and the bounds for $|f^{(n)}(z)|$.

OR

Q 5.
(a) Let $f : B(0,1) \to B(0,1)$ be analytic, satisfying $f(1/2) = 3/4$. What is the maximum possible value of $|f'(1/2)|$.
(b) Let $g : B(0,1) \to B(0,1)$ be analytic, satisfying $g(0) = 1/2$. Find, if there exists such a g such that $g'(0) = 3/4$.
Use Schwarz-Pick.

Q 6. Let $f : B(0,1) \to \mathbb{C}$ be analytic, satisfying $f(0) = 0, f'(0) = 1, |f(z)| \leq M$ for all $z \in B(0,1)$. Prove that $f(B(0,1)) \supseteq B(0,1/6M)$.
Hint. Use Cauchy estimate to observe, on the circle $C(0,1/4M)$, that $|f(z)| \geq 1/6M$. For $w \in B(0,1/6M)$, show $w \in Im(f)$ by applying Rouche's theorem to $f(z)$ and $f(z) - w$.

OR

Q 6. Let f, g be meromorphic functions such that $f^n + g^n$ is the constant function 1 for some $n > 2$. Prove that either f, g are constant functions or they have the some common poles. Give an example of non-constant f, g as above for $n = 4$.
Hint. Apply Little Picard for a suitable meromorphic function.

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Q 7. Prove that $\int_{\mathbb{R}} \frac{x^2 dx}{(x^2+4)^2(x^2+9)} = \frac{\pi}{100}$.

OR

Q 7. Prove that $\int_0^\infty \frac{\log(x)dx}{(1+x)^3} = \frac{-1}{2}$.
Hint. Consider the path integral of $\log(z)^2/(1+z)^3$ over a key-hole contour avoiding 0 and containing -1.